

# Series Solutions of the N-Dimensional Position-Dependent Mass Schrödinger Equation with a General Class of Potentials

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## Abstract

The analytical solutions of the  $N$ -dimensional Schrödinger equation with position-dependent mass for a general class of central potentials is obtained via the series expansion method. The position-dependent mass is expanded in series about origin. As a special case, the analytical bound-state series solutions and the recursion relation of the linear-plus-Coulomb (Cornell) potential with the decaying position-dependent mass  $m = m_0 e^{-\lambda r}$  are also found.

Keywords: Cornell potential, Position-dependent mass, Series expansion method

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## I. INTRODUCTION

The solution of the Schrödinger equation with position-dependent mass for any spherically symmetric potential has attracted attention over the past years [1-17]. The motivation in this direction arises from considerable applications in the different fields of the material science

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and condensed matter physics. For instance, such applications in the case of the bound states in quantum system [4], the nonrelativistic Green's function for quantum systems with the position-dependent mass [5], the Dirac equation with position-dependent mass in the Coulomb field [5], electronic properties of semiconductors [10],  $^3He$  cluster [11], quantum dots [12], semiconductor heterostructures [13,14], quantum liquids [15], the dependence of energy gap on magnetic field in semiconductor nano-scale quantum rings [16], the solid state problems with the Dirac equation [17]. Almost all of those works mentioned above were focused on obtaining the energy eigenvalues and the potential function for the given quantum system with the position-dependent mass. The wave functions were either obtained by the solutions to the Schrödinger equation with the constant mass, or a few lower excited states were obtained by acting of the creation operator on the ground state. The effective potentials are the sum of the real potential form and the modification terms emerged from the location dependence of the effective mass [2]. Taking into consideration the position-dependent mass, the aim of this work is to carry out the analytical solutions of the  $N$ -dimensional Schrödinger equation with position-dependent mass for a general class of static quarkonium potentials by the series expansion method used in [18,19]. Additionally, we investigate the linear-plus-Coulomb (Cornell) potential case [20,21].

The contents of this paper is as follows. In Section II, we present the  $N$  dimensional Schrödinger equation with position-dependent mass for any spherically symmetric potential. In Section III the analytical bound-state series solutions of a general class of quarkonium potentials. In Section IV, we study the analytical series solutions of the Cornell potential for a particle of an exponentially decaying mass,  $m = m_0 e^{-\lambda r}$ ,  $\lambda > 0$ , case and then give an example for the analytical calculations of the Coulomb's wave function. Finally, in Section V we give our concluding remarks.

## II. THE $N$ -DIMENSIONAL POSITION-DEPENDENT MASS SCHRÖDINGER EQUATION

The wave Schrödinger equation with position-dependent mass for a spherically symmetric potential  $V(r)$  in  $N$ -dimensional space (in  $\hbar = 1$  units):

$$\nabla_N \frac{1}{m} \nabla_N \psi(r) + 2[E - V(r)] \psi(r) = 0, \quad (1)$$

where the wave function is defined by [20,21,22,23]

$$\psi(r) = e^{-(N-1)/2} R_{n,l}(r) Y_{l,m}(x). \quad (2)$$

and  $m = m(r)$ . We substitute  $\nabla_N \frac{1}{m(r)} \nabla_N \psi(r) = \left( \nabla_N \frac{1}{m(r)} \right) \cdot (\nabla_N \psi(r)) + \frac{1}{m(r)} \nabla_N^2 \psi(r)$  into Eq.(1) and obtain the  $N$ -dimensional position-dependent mass radial Schrödinger equation

$$\left\{ \frac{d^2}{dr^2} + \frac{m'}{m} \left( \frac{N-1}{2r} - \frac{d}{dr} \right) - \frac{[k-1][k-3]}{4r^2} + 2m(r)[E - V(r)] \right\} R_{n,l}(r) = 0, \quad (3)$$

where  $k = N + 2l$  and  $m'(r) = dm/dr$ . It can be clearly seen that for  $m' = 0$  case, the above equation reduces to the well known equation with constant mass used in Refs.[20,21,22,23].

In the present work, we are concerned in bound states, i.e.,  $E < 0$ . On the other hand, one should be careful about the behavior of the wave function  $R(r)$  near  $r = 0$  and  $r \rightarrow \infty$ . It may be mentioned that  $R(r)$  behaves like  $r^{(k-1)/2}$  near  $r = 0$  and it should be normalizable. We choose the wave function [18,19,24,25]

$$R_{n,l}(r) = r^{(k-1)/2} e^{-br} u(r), \quad b = \sqrt{-2m_0 E}, \quad (4)$$

where  $m_0$  is the initial value of mass. It should be noted that  $N$  and  $l$  enter into expression (3) in the form of the combination  $k = N + 2l$ . Consequently, the solutions for a particular central potential  $V(r)$  are the same as long as  $k$  remains unaltered. Thus, the  $s$ -wave eigensolution ( $R_{n0}$ ) and eigenvalues ( $E$ ) in four-dimensional space are identical to the  $p$ -wave two-dimensional solutions. By substituting Eq.(4) into Eq.(3), we obtain

$$\left[ (k-1) \frac{d}{dr} - (k-1)b - \frac{m'}{m} l \right] u(r) +$$

$$\left[ \frac{d^2 u(r)}{dr^2} - \left( 2b + \frac{m'}{m} \right) \frac{du(r)}{dr} + \frac{m'}{m} b u(r) - 2mV(r)u(r) + 2E(m - m_0)u(r) \right] r = 0. \quad (5)$$

Next we consider a series solutions for the above reduced radial wave Schrödinger equation.

### III. THE SERIES SOLUTION WITH A CLASS OF STATIC POTENTIALS

Equation (3) cannot be solved exactly except for some particular cases. Nevertheless, we can get an approximate solution using the series expansion method [18,19]. In application, we consider here a group of central potentials belong to the following general form [23]

$$V(r) = -V_1 r^{-\alpha} + V_2 r^\beta + V_3, \quad (6)$$

where  $V_1$  and  $V_2$  are positive coupling constants whereas the constant  $V_3$  may be of either sign. Moreover, this group of potentials satisfies the boundary conditions stated in [20,21,22,23]. This class of generality for potentials (6) is used to produce the bound state energy spectra for quarkonium systems [20,21,22,23]. It comprises a well-known potential, e.g., the Cornell potential (we set  $\alpha = \beta = 1$ ,  $V_1 = A = \frac{4}{3}\alpha_s$ ,  $V_2 = B$ ,  $V_3 = C$ ) (cf. Ref.[21]).

Now we try the following series expansions about origin:

$$m(r) = \sum_{\nu=0}^{\infty} b_\nu r^\nu = b_0 + \sum_{\nu=1}^{\infty} b_\nu r^\nu, \quad b_0 = m_0 \quad (7)$$

$$\frac{m'}{m} = \sum_{\nu=0}^{\infty} b'_\nu r^\nu, \quad (8)$$

together with the series expansion for  $u(r)$

$$u(r) = \sum_{i=0}^{\infty} a_i r^i, \quad a_0 \neq 0 \quad (9)$$

and substitute Eqs.(6)-(9) into Eq.(5), we obtain the following relation

$$\begin{aligned} & \sum_{i=0}^{\infty} \left[ -(k-1)ba_i r^i - 2bia_i r^i + (k-1)ia_i r^{i-1} + i(i-1)a_i r^{i-1} \right. \\ & \left. - \sum_{\nu=0}^{\infty} b'_\nu \left( ia_i r^{\nu+i} + la_i r^{\nu+i} - ba_i r^{\nu+i+1} \right) + b^2 a_i r^{i+1} + 2E \sum_{\nu=0}^{\infty} b_\nu a_i r^{\nu+i+1} \right] \end{aligned}$$

$$+2V_1 \sum_{\nu=0}^{\infty} b_{\nu} a_i r^{\nu+i-\alpha+1} - 2V_2 \sum_{\nu=0}^{\infty} b_{\nu} a_i r^{\nu+i+\beta+1} - 2V_3 \sum_{\nu=0}^{\infty} b_{\nu} a_i r^{\nu+i+1} \Big] = 0, \quad (10)$$

with  $b^2 = -2m_0 E$ . On the other hand, we define

$$M_i = \sum_{j,\nu=0}^{j+\nu=i} a_j b_{\nu}; \quad i \geq 0 \quad (11)$$

$$M'_i = \sum_{j,\nu=0}^{j+\nu=i} a_j b'_{\nu}; \quad i \geq 0 \quad (12)$$

$$T_i = \sum_{j,\nu=0}^{j+\nu=i} j a_j b'_{\nu}; \quad i \geq 1, \quad T_0 = 0. \quad (13)$$

Setting the coefficients of the power of  $r^n$  to be zero, we obtain the following recurrence relation of the bound energy spectrum

$$\begin{aligned} n(n+1)a_{n+1} + (k-1)(n+1)a_{n+1} - (k-1)ba_n - 2bna_n - lM'_n + bM'_{n-1} - T_n \\ + 2EM_{n-1} + b^2a_{n-1} + 2V_1M_{n+\alpha-1} - 2V_2M_{n-\beta-1} - 2V_3M_{n-1} = 0, \end{aligned} \quad (14)$$

with the final radial wave functions

$$R_{n,l}(r) = r^{(k-1)/2} e^{-br} \sum_{i=0}^{\infty} a_i r^i, \quad a_0 \neq 0. \quad (15)$$

We now present some special cases.

### A. Coulomb Potential

For the solutions of the Coulomb problem in  $N$ -dimensional space, we set  $V_1 = Z$ ,  $V_2 = V_3 = 0$ ,  $\alpha = 1$ ,  $\beta = 0$ , then we get its recurrence relation from Eq.(14):

$$\begin{aligned} n(n+1)a_{n+1} + (k-1)(n+1)a_{n+1} - (k-1)ba_n - 2bna_n \\ - lM'_n + bM'_{n-1} - T_n + 2EM_{n-1} + b^2a_{n-1} + 2ZM_n = 0, \end{aligned} \quad (16)$$

with the radial wave functions are given in Eq.(15), cf. Ref.[24]. This case was treated in Ref.[19].

### B. Harmonic Oscillator Potential

For the solutions of the harmonic oscillator problem in  $N$ -dimensional space, we set  $V_2 = \omega^2$ ,  $V_1 = V_3 = 0$ ,  $\alpha = 0$ ,  $\beta = 2$ , then the recurrence relation (14) becomes

$$n(n+1)a_{n+1} + (k-1)(n+1)a_{n+1} - (k-1)ba_n - 2bna_n - lM'_n + bM'_{n-1} - T_n + 2EM_{n-1} + b^2a_{n-1} - 2\omega^2M_{n-3} = 0, \quad (17)$$

with the radial wave functions are given in Eq.(15). This case was also treated in Ref.[19].

### C. Confining Linear Potential

For the solutions of the confining linear potential in  $N$ -dimensional space, we set  $V_1 = V_3 = 0$ ,  $V_2 = B$ ,  $\alpha = 0$ ,  $\beta = 1$ , then Eq.(14) becomes

$$n(n+1)a_{n+1} + (k-1)(n+1)a_{n+1} - (k-1)ba_n - 2bna_n - lM'_n + bM'_{n-1} - T_n + 2EM_{n-1} + b^2a_{n-1} - 2BM_{n-2} = 0, \quad (18)$$

with the radial wave functions are still given in Eq.(15).

### D. Cornell Potential

Now we investigate a confinement potential consisting of an attractive Coulomb term and a confining linear potential used for calculation of quarkonium ( $q\bar{q}$ ) bound-state masses [20,21]. With the set of parameters  $V_1 = A = 4\alpha_s/3$ ,  $V_2 = B$ ,  $V_3 = C$ ,  $\alpha = \beta = 1$ , then the recurrence relation (14) becomes

$$n(n+1)a_{n+1} + (k-1)(n+1)a_{n+1} - (k-1)ba_n - 2bna_n - lM'_n + bM'_{n-1} - T_n + 2EM_{n-1} + b^2a_{n-1} + 2AM_n - 2BM_{n-2} - 2CM_{n-1} = 0, \quad (19)$$

with the radial wave functions are given in Eq.(15).

Let us investigate the last case. The recurrence relation (19) implies

$$a_1 = \left[ b + \frac{lb'_0 - 2Am_0}{k-1} \right] a_0, \quad (20)$$

$$a_2 = \frac{[(k+1)b + (l+1)b'_0 - 2Am_0]}{2k} a_1 + \frac{[lb'_1 - bb'_0 - 2Ab_1 + 2Cm_0]}{2k} a_0, \quad (21)$$

$$a_3 = \frac{[(k+3)b + (l+2)b'_0 - 2Am_0]}{3(k+1)} a_2 + \frac{[(l+1)b'_1 - bb'_0 - 2Ab_1 + 2Cm_0]}{3(k+1)} a_1 \\ + \frac{[lb'_2 - bb'_1 + 2(C-E)b_1 - 2Ab_2 + 2Bm_0]}{3(k+1)} a_0. \quad (22)$$

#### IV. THE ANALYTICAL SERIES SOLUTIONS FOR $M = M_0 E^{-\lambda R}$

We choose one simple example considered recently by Ref.[18]. Assuming a particle with an exponentially decaying position-dependent effective mass  $m(r) = m_0 e^{-\lambda r}$ ,  $\lambda > 0$  [18]. This form is taken on the base that the position-dependent mass must be convergent when  $r \rightarrow \infty$ . We try the series expansion

$$m(r) = \sum_{\nu=0}^{\infty} m_{\nu} r^{\nu}, \quad m_0 = 1, \quad (23)$$

and

$$\frac{m'}{m} = -\lambda. \quad (24)$$

Thus, for Cornell potential, Eq.(5) gives the following recursion relation

$$[(k-1)n + n(n-1)] a_n - [b(k-1) + (2b-\lambda)(n-1) - l\lambda] a_{n-1} \\ + b(b-\lambda) a_{n-2} + 2Em_{n-2} a_{n-2} + 2Am_{n-1} a_{n-1} - 2Bm_{n-3} a_{n-3} - 2Cm_{n-2} a_{n-2} = 0, \quad (25)$$

which implies

$$a_1 = \left[ b - \frac{l\lambda + 2Am_0}{k-1} \right] a_0, \quad (26)$$

$$a_2 = \frac{[(k+1)b - \lambda(l+1) - 2Am_0]}{2k} a_1 + \frac{[2Cm_0 + \lambda b - 2Am_1]}{2k} a_0, \quad (27)$$

$$a_3 = \frac{[(k+3)b - (l+2)\lambda - 2Am_0]}{3(k+1)} a_2 + \frac{[\lambda b + 2Cm_0 - 2Am_1]}{3(k+1)} a_1 + \frac{[2Bm_0 + 2(C-E)m_1 - 2Am_2]}{3(k+1)} a_0. \quad (28)$$

Finally, for another application of the model in case of a constant particle mass  $m = m_0$ . We find the Coulomb's wave functions in three-dimensional space,  $N = 3$ , easily through the relations (26)-(28). We present the following results [24]

$$a_1 = -\frac{2Am_0}{(n+l+1)} \frac{n}{(2l+2)} a_0, \quad (29)$$

$$a_2 = \left( -\frac{2Am_0}{n+l+1} \right)^2 \frac{n(n-1)}{2!(2l+3)(2l+2)} a_0, \quad (30)$$

$$a_3 = \left( -\frac{2Am_0}{n+l+1} \right)^3 \frac{n(n-1)(n-2)}{3!(2l+4)(2l+3)(2l+2)} a_0, \quad (31)$$

and from which the wave functions is given by

$$R_{n,l}(r) = r^{l+1} e^{-\frac{Am_0}{(n+l+1)}r} \sum_{i=0}^{\infty} (-1)^i \left( \frac{2Am_0}{n+l+1} \right)^i \frac{(2l+1)!n!}{(2l+1+i)!(n-i)!i!} a_0, \quad (32)$$

with a proper normalization

$$a_0 \approx N_{n,l}^{(C)} = \left[ \frac{2Am_0}{n+l+1} \right]^{l+1} \frac{1}{(n+l+1)} \sqrt{\frac{Am_0 n!}{(n+2l+1)!}}, \quad n = 0, 1, 2, \dots \quad (33)$$

On the other hand, considering a particle with an exponentially decaying position-dependent effective mass  $m(r) = m_0 e^{-\lambda r}$ , the relations (26)-(28) read [24]

$$a_1 = \frac{Am_0}{(l+1)} \left[ \frac{l+1}{n+l+1} - \frac{l\lambda}{2m_0 A} - 1 \right] a_0, \quad (34)$$



$$a_2 = \frac{Am_0}{(2l+3)} \left\{ \left[ \frac{l+2}{n+l+1} - \frac{(l+1)\lambda}{2Am_0} - 1 \right] a_1 + \left[ \frac{\lambda}{2(n+l+1)} - \frac{m_1}{m_0} \right] a_0 \right\}, \quad (35)$$

$$a_3 = \frac{Am_0}{3(l+2)} \left\{ \left[ \frac{l+3}{n+l+1} - \frac{(l+2)\lambda}{2Am_0} - 1 \right] a_2 + \left[ \frac{\lambda}{2(n+l+1)} - \frac{m_1}{m_0} \right] a_1 - \left[ \frac{Am_1}{2(n+l+1)} - \frac{m_2}{m_0} \right] a_0 \right\}, \quad (36)$$

and from which the Coulomb's wave functions

$$R_{n,l}(r) = r^{l+1} e^{-\frac{Am_0}{(n+l+1)}r} \left[ a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \dots \right]. \quad (37)$$

where  $a_i$ , with  $i = 0, 1, 2, 3$ , are given by means of Eqs.(33)-(36).

## V. CONCLUDING REMARKS

We have carried out the series solutions of position-dependent mass  $N$ -dimensional radial Schrödinger equation for a general class of static potentials which are mostly used in the calculations of quarkonium bound-state masses. Some interesting results including the Coulomb, linear, harmonic, and Cornell potentials are also considered. In particular, the Cornell case was investigated and the series solution for the  $N$ - dimensional radial Schrödinger equation with an exponentially decaying position-dependent effective mass  $m(r) = m_0 e^{-\lambda r}$ , are solved analytically. Contrary to most of the approximation methods which are valid when the parameters are small, our results obtained by this approximation method are also valid with large values of the parameters. These analytical calculations can be applied to some related fields of physics such as quarkonium systems in three dimensional space, i.e.,  $N = 3$ . The work presented through the two examples discussed above would appear to be of interest in the ground that it offers an explicit formulas used to construct the wave functions for a central potential [20,21,22,23,24].

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